General and math-specific predictors of sixth-graders' knowledge of fractions

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The present study examined predictors of student's knowledge of fraction concepts and procedures in sixth grade (N=334). Predictors included both math-specific and more general competencies, which were assessed in fifth grade. Multiple regression analyses showed that whole number line estimation, non-symbolic proportional reasoning, long division, working memory, and attentive behavior contributed uniquely to a general measure of students' fraction concepts; on a measure of fraction procedures, whole number line estimation, multiplication fact fluency, division, and attention made unique contributions. The combined predictability of the measures was lower for fraction procedures than for fraction concepts. Although the unique predictors and the amount of explained variance differed according to the fraction outcome, the ability to locate whole numbers on the number line was a major contributor to prediction in each model. Non-symbolic proportional reasoning was particularly predictive of children's conceptual understanding of fractions.

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Facility with fractions supports student learning of advanced mathematics, such as algebra and geometry (National Mathematics Advisory Panel [NMAP], 2008). Fraction knowledge in middle school accounts for much of the gains students make in mathematics achievement more generally (Bailey, Hoard, Nugent, & Geary, 2012). Additionally, fraction knowledge at ten years of age predicts algebra knowledge in high school, after controlling for family education, income, IQ, and knowledge of whole number arithmetic (Siegler et al., 2012). Proficiency in algebra and advanced mathematics, in turn, prepares students for success in higher education as well as for careers involving science, technology, engineering, and mathematics (NMAP, 2008; Sadler & Tai, 2007). There are many potential sources of students’ difficulties with fractions. For example, some students may attend separately to the numerator or the denominator, rather than considering the magnitude that the fraction represents (Meert, Grégoire, & Noël, 2008). Moreover, the “rules” students learn in early elementary school for whole numbers do not always apply to fractions (Ni & Zhou, 2005). Multiple fractions can refer to the same magnitude (4/8 and 6/12). Fractions can be represented in various ways, even though they have one and only one precise mathematical definition. They can be understood as a part of a whole, a division problem, a ratio, an operator, or a measure on a number line (Behr & Post, 1992). Many students have trouble developing core fraction concepts for these reasons. Around the same time students are learning fraction concepts, they must also engage in arithmetic computations with fractions, which are different from computation with whole numbers (e.g., multiplication of fractions leads to a smaller number). Skill development in fraction concepts and fraction procedures is intertwined, with an increase in one type of knowledge leading to an increase in knowledge of the other (Hecht, Close, & Santisi, 2003; Hecht & Vagi, 2010, 2012; Rittle-Johnson & Siegler, 1998; Siegler et al., 2012). Thus, development of both fraction concepts and procedures are important areas of concern.

Fractions are complex, and not surprisingly, individual differences in a range of number-related and more general processes affect learning of fraction concepts and procedures (Hecht & Vagi, 2012; Hecht, Vagi, & Torgesen, 2007; Jordan et al., 2013). In the present study, we investigated the degree to which a theoretically motivated group of number-related competencies (e.g., whole number line estimation, non-symbolic proportional reasoning, multiplication fluency, and division), as well as more general competencies in working memory, attentive behavior, and reading fluency predict fractions outcomes at the end of sixth grade. Sixth grade is a key benchmark period for examining competence with fractions, as students typically have had roughly three years of formal instruction on the topic; this instructional trajectory is reflected in the Common Core State Standards in Mathematics (Council of Chief State School Officers and National Governors Association Center for Best Practices, 2010). Moreover, sixth grade often is the last year students receive instruction with an intensive focus on fractions, and the gap between students with low fractions achievement and students with high fraction achievement in sixth grade widens by eighth grade (Siegler & Pyke, 2013).

1. Number-related predictors of fraction knowledge

1.1. Numerical magnitude representations

Accurate representations of magnitudes on a mental number line are important for the early acquisition of fraction skills and knowledge, just as they are for whole number skill acquisition (Siegler, Thompson, & Schneider, 2011). Jordan et al. (2013) found that students’ ability to estimate whole number locations on a 0–1000 number line in third grade uniquely predicted their performance on fraction outcomes in fourth grade, over and above other number-related and domain-general skills. Similarly, Vukovic et al. (2014) report that number line estimation in second grade predicts fraction concepts in fourth grade.

One explanation for this finding is that students who have a strong grasp of whole number magnitudes can more easily make the connection that fractions, too, have magnitudes and can be absolute measures of quantity (e.g., 1/2 min is the same as 30 s). An alternate explanation is that estimating numbers on a number line may involve proportional reasoning. Thinking about a number line in terms of its constituent parts enhances children’s accuracy on estimation tasks (Barth & Paladino, 2011). For example, 1000 can be divided in half or into quarters to estimate locations on the line (e.g., 245 would be close to the quarter reference of 250).
The ability to locate fractions on number lines (e.g., 1/7 on a 0–2 number line) is likely to be an important aspect of fraction learning (Bailey et al., 2012; Mazzocco & Devlin, 2008; Siegler et al., 2011). Knowledge of fraction magnitudes seems to indicate that students have overcome the “whole number bias” that occurs when whole number knowledge is inappropriately generalized to fractions concepts learned later in schooling (e.g., Hiebert & Wearne, 1986; Ni & Zhou, 2005; Rittle-Johnson & Siegler, 1998; Schneider & Siegler, 2010; Siegler et al., 2011). That is, students who have a strong understanding of fraction magnitudes are better able to recognize that the entire fraction represents a single magnitude, rather than looking at only the numerator or denominator of the fraction to determine which is bigger.

1.2. Non-symbolic proportional reasoning

Non-symbolic representations provide a foundation for understanding symbolic numbers (Siegler & Lortie-Forgues, 2014). Non-symbolic proportional reasoning, which involves thinking about parts and wholes, is associated with fraction learning. Students must understand equivalence of ratios expressed as fractions (i.e., \(a/b = c/d\)) (Behr, Harel, Post, & Lesh, 1992; Boyer, Levine, & Huttenlocher, 2008; Watson, Beswick, & Brown, 2012). For example, six pens for three people are proportionally the same as 12 pens for six people (2:1). The ability to “see” multiplicative relationships between these equivalent fractional quantities may underpin success with fractions (Boyer & Levine, 2012). Tasks that require greater scaling up from a visually depicted, equivalent target proportion are harder for most children than ones that require less scaling (e.g., 1/4 to 4/16 is harder than 1/4 to 2/8). Greater scaling requires larger transformations (Boyer & Levine, 2012; Vasilyeva & Huttenlocher, 2004; Vasilyeva, Duffy, & Huttenlocher, 2007). Not surprisingly, Möhring, Newcombe, Levine, and Frick (2013) report a strong correlation between non-symbolic proportional reasoning, as measured by the child’s ability to judge the proportional equivalence of visually depicted proportions, and performance on conventional fraction comparison tasks (e.g., “What fraction is smaller, 4/12 or 2/12?”). Möhring et al. suggest that performance on the proportional reasoning task reflects a kind of “spatial sense” that helps children estimate plausibility when comparing fractions. However, the unique contribution of proportional reasoning on a non-symbolic visual task within a set of other relevant predictors of fraction knowledge has not been investigated.

1.3. Long division

Whole number division and fractions are closely related both mathematically and in terms of student performance (Siegler & Pyke, 2013; Siegler et al., 2012). Fractions are essentially a form of division (\(a/b\)), and division is the only whole number operation that can yield fractional answers. The traditional long division algorithm requires students to integrate several other arithmetic operations and remember multiple steps, as do advanced fractions operations (e.g., addition of fractions with unlike denominators). Because of these relations, we hypothesized that facility with whole number long division has a uniquely important role in the development of fraction knowledge. The relative importance of long division to fraction development is potentially important, as division concepts are often underemphasized in school (Siegler et al., 2012).

1.4. Fact fluency

Fluent knowledge of single number arithmetic facilitates application of more complex mathematics procedures (Locuniak & Jordan, 2008). Poor fluency with whole number arithmetic characterizes students with mathematics learning disabilities (Geary, 2004; Jordan & Hanich, 2000; Jordan, Hanich, & Kaplan, 2003) and appears to extend to fraction learning (Hecht & Vagi, 2010; Jordan et al., 2013). In the present study, we specifically examined the role of multiplication fact fluency in fraction performance, because facility with multiplication, in particular, helps students carry out fraction procedures (e.g., cross-multiplying) and quickly find the least common denominator in fractions problems. We
predicted that multiplication fact fluency would contribute independently to fraction knowledge – especially fraction procedures – over and above other mathematics-related skills.

2. More general predictors of fraction knowledge

Cognitive factors beyond number-related skills also contribute to fraction learning (e.g., Hecht et al., 2003; Jordan et al., 2013; Vukovic et al., 2014). In the present study, we included measures of two general processes that are strongly related mathematics learning more generally (Fuchs et al., 2005, 2006; Geary, 2004) and to fraction development in particular (Hecht & Vagi, 2010; Hecht et al., 2003): working memory and attention.

2.1. Working memory

Working memory involves the ability to store and manipulate information in short-term memory. The importance of working memory in mathematics is well-established (e.g., Fuchs et al., 2010; Geary, 2011; Geary, Hoard, Byrd-Craven, Nugent, & Numtee, 2007). Working memory, as measured by the ability to recall a series of count sequences, uniquely predicts mathematics outcomes in the early elementary grades (Fuchs et al., 2010; Meyer, Salimpoor, Wu, Geary, & Menon, 2010). In the early stages of fraction learning, Jordan et al. (2013) found that working memory on a counting recall task in third grade was a small predictor (among a series of tasks) of fourth graders’ knowledge of fraction procedures but it did not predict fraction concepts. Hecht et al. (2003) suggest that working memory predicts arithmetic knowledge, which in turn influences skill with fraction computations. The present study sought to investigate whether the relation between working memory and fraction outcomes becomes more important in later grades, as fraction procedures become more complex and the demands for holding and manipulating numerical information increase.

2.2. Attention

Attention allows students to focus on mathematics instruction and stay on task to acquire fraction-relevant knowledge (Finn, Pannozzo, & Voelkl, 1995). Attentiveness in classrooms relates to a wide variety of mathematical outcomes, such as accuracy in recalling facts, executing computational procedures, and solving story problems (e.g., Fuchs et al., 2005). Children’s attentive behavior also emerges as a strong, unique predictor of fraction concepts and procedures in fourth grade, second only to number line estimation accuracy with whole numbers (Jordan et al., 2013). Students who are attentive in their mathematics classrooms develop a better conceptual understanding of fractions, which in turn positively influences their ability to solve fraction computations (Hecht et al., 2003). It is likely that this relation will remain important for fraction learning in subsequent grades.

3. The present study

The present study extends findings from Jordan et al. (2013) who examined third-grade cognitive predictors of fraction learning at the end of fourth grade, in the beginning stages of instruction. In sum, the researchers found that number line estimation, addition skill, and attention contributed uniquely to fraction concepts and procedures; working memory contributed uniquely to knowledge of fraction procedures but not to fraction concepts. Number line estimation on a 0–1000 number line made the largest independent contribution to both kinds of fraction knowledge. Interestingly, a non-symbolic approximate number system (ANS) magnitude comparison task was not related to either fraction outcome.

The aim of the present study was to look at predictors of sixth-grade knowledge of fraction concepts and fraction procedures using a longitudinal data set. All of the predictor measures were given in fifth grade. We were especially interested in seeing whether numerical magnitude understanding continues to be strongly predictive of fraction knowledge in middle school – relative to other domain-specific and general cognitive predictors – as fraction learning becomes more complex. We also looked at student’s performance on a non-symbolic proportional reasoning task to see the contribution of a more
fundamental visual/spatial reasoning task that does not involve numerical symbols. The non-symbolic proportional reasoning task is likely be more relevant than an ANS task, because there is evidence that proportional reasoning may be underlying children’s informal understandings of fractions before formal fraction instruction (Mix, Levine, & Huttenlocher, 1999). Additional number-related predictors were whole number long division, which typically is taught in fifth grade, and whole number multiplication fluency. Working memory and attentive behavior also were included in our models as well as a measure of reading fluency (to determine whether fluency is predictive of fraction outcomes more generally). Dependent variables, assessed a year later in sixth grade, included measures of fraction concepts (e.g., interpreting the area or part-whole model for representing fractions, equivalence, and fraction number line estimation) and fraction procedures (e.g., addition, subtraction, multiplication, and division of fractions).

4. Method

4.1. Participants

Participants were fifth graders from nine elementary schools in two public school districts serving families of diverse socioeconomic status. We followed students as they entered sixth grade (middle school), and to the extent possible, continued assessing students who moved out of the school districts. Students were part of a larger longitudinal study of children’s mathematical development. Only students who completed all of the predictor measures and outcome measures described below were included (N = 334 out of 409). Little’s (1988) missing completely at random (MCAR) test results were not significant ($\chi^2 = 105.162; df = 101; p = .369$), indicating data were missing completely at random. Demographic information for study is shown in Table 1.

4.2. Predictor measures

4.2.1. Whole number line estimation

Children estimated the position of whole numbers on a number line that ranged from 0 to 1000 (Siegler & Opfer, 2003). Twenty-two numerals (56, 606, 179, 122, 34, 78, 150, 938, 100, 163, 754, 5, 725, 18, 246, 722, 818, 738, 366, 2, 486, and 147) were presented individually (see Fig. 1 for an item example). Each numeral was shown below the center of the line. The items were presented on a laptop computer using DirectRT v2012 and scored electronically. Children responded by moving the cursor...
to where they thought the number was located on the number line, and pressing a key to indicate their answer. Then, a new number appeared and the process repeated. After each item, the cursor was repositioned to “0” on the number line to discourage students from using their response to the previous item as an anchor for the next item. To introduce students to the task, the assessor first asked the student to practice moving the cursor and then modeled where a number (270) should go on the number line. The measure had high internal consistency (α = .91).

The score on this task was calculated as the absolute value of the difference between the estimated position and actual position, divided by the numerical range of the number line (1000), multiplied by one hundred. A higher percent absolute error (PAE) score indicates poorer performance on the task.

4.2.2. Non-symbolic proportional reasoning

The equivalence task was adapted from Boyer and Levine (2012). At the beginning of the task, the assessor introduced the student to a character named “Harry the Hog” and explained that Harry only likes to drink juice that tastes “just right.” For each trial, Harry appeared on the upper left corner of the computer screen. The target juice mixture appeared as a narrow vertical bar with red juice parts and blue water parts. To the right of the target were two choice alternatives – one with the proportions equivalent match to those of the target juice mixture, and the other an incorrect match. At the beginning of each trial, the assessor pointed to the target bar and the two alternatives while asking the child, “Which of these two is the right mix for the juice Harry is trying to make?” The goal was to pick the juice that would taste just like Harry’s juice. An example item illustrating both the continuous and discrete condition is shown in Fig. 2. There were a total of 48 self-paced test trials and one practice trial. The items were presented in a random order determined by the computer. Half of the problems had discrete markings indicating each part and the other half showed continuous quantities (no markings). The correct answer for each problem involved either scaling down from a larger target to a smaller match (e.g., 3/9 to 2/6) or scaling up from a smaller target to a larger match (e.g., 2/6 to 3/9). Each trial included a target mixture that differed from the equivalent choice mixture by one of the following scaling factors: 0.67, 0.50, 0.33, 3.00, 2.00, or 1.50. Internal reliability was (α = .93) in fifth grade.

4.2.3. Whole number long division

Students were given six long division problems (presented traditionally with the dividend inside the long division symbol and the divisor to the left) and asked to solve as many as they could in eight minutes (56 ÷ 8, 42 ÷ 3, 306 ÷ 9, 91 ÷ 4, 180 ÷ 60, and 1400 ÷ 400). Internal reliability (α) of this measure was .76.

4.2.4. Multiplication fact fluency

The multiplication fluency subtest of the Wechsler Individual Achievement Test (WIAT; The Psychological Corporation, 1992) was used. Students have one minute to solve 40 written
multiplication problems in which the multiplicands are between 0 and 10. The raw score is the number of problems solved correctly. Test–retest reliability in fifth grade is .90.

4.2.5. Working memory
To assess working memory, the Counting Recall subtest of the Working Memory Test Battery for Children (WMTB-C; Pickering & Gathercole, 2001) was used. The child was asked to count collections of scattered red dots printed on individual cards and then to recall the number of dots that were counted on each card. The number of cards in a series ranged from one to seven. For example, for a series of three cards the child might count 4 dots, 6 dots, and then 7 dots. At the end of the series, the child would then have to recall how many were counted on each card in the correct order. For each series, there were six trials. The child needed to answer correctly on at least 3 trials to move on to the next one. The total possible score range was from 0 to 42. Test-retest reliability of this measure is .61.

4.2.6. Attentive behavior
Teachers’ responses on the inattention subscale of the SWAN Rating Scale (Swanson et al., 2006) were used to measure children’s attention in the classroom. The scale consists of nine items that are based on the criteria for Attention Deficit Hyperactivity Disorder for inattention from the Diagnostic and Statistical Manual of Mental Disorders-IV (APA, 1994). Teachers were asked to rate each child’s attention during math class, relative to other children of the same age, on a scale from 1 to 7 (below average to above average attention). Scores range between 9 (a score of 1 on each of the 9 items) and 63 (a score of 7 on each of the 9 items). This instrument has been used in previous research in mathematics (Fuchs et al., 2006, 2010; Vukovic et al., 2014). Internal reliability of the measure is high (α > .97).

4.2.7. Reading fluency
The Sight Word Efficiency subtest of the Test of Word Reading Efficiency (TOWRE; Torgesen, Wagner, & Rashotte, 1999) was used to assess reading fluency. Children were given 45 s to read aloud as many words from a list as possible. The raw score was the number of words read correctly. Test–retest reliability in fifth grade is .92.
4.3. **Outcome measures**

4.3.1. **Fraction concepts**

Fraction concepts contained items involving interpreting the area or part-whole model for representing fractions, magnitude comparisons, equivalence, and estimation. There were three shaded fraction items from Hecht, Close, and Santisi (2003) and 25 fraction items released from recent National Assessments of Educational Progress (NAEP, 2007, 2009). For the shaded fraction items, students shaded a figure, or a set of figures indicating the amount represented by a fraction symbol. The NAEP items assess part-whole understanding (e.g., “The figure above shows that part of a pizza has been eaten. What part of the pizza is still there?”), estimation (e.g., the student is shown four fractions and asked, “Which fraction has a value closest to 1/2?”), and fraction comparison and equivalence (e.g., “Which picture shows that 3/4 is the same as 6/8?”). The total possible fraction concepts score was 28, and internal reliability was .86 in sixth grade.

4.3.2. **Fraction number line estimation**

We also assessed knowledge of numerical magnitudes specifically. Children estimated where written fractions should go on 0–1, 0–2, and 0–5 number lines (adapted directly from Siegler et al., 2011). The procedures were identical to those used for whole number estimation, described above. To acquaint students with the procedure, the assessor modeled where a fraction (1/8) should go on the 0–1 number line, and then asked the student to locate a practice fraction (1/4). No feedback was given. Students estimated the positions of the following fractions on the 0–1 number line: 1/5, 13/14, 2/13, 3/7, 5/8, 1/3, 1/2, 1/19, and 5/6.

Next, students were given fractions to estimate on a 0–2 number line. Here, the assessor modeled two fractions (1/8 and 1 1/8). Students were asked to locate a practice fraction (1/4) but no feedback was given. 0–2 number line test items included both proper and improper fractions, as well as mixed numbers: 1/3, 7/4, 12/13, 1 11/12, 3/2, 5/6, 5/5, 1/2, 7/6, 1 2/4, 1, 3/8, 1 5/8, 2/3, 1 1/5, 7/9, 1/19, 1 5/6, and 4/3. The same procedure was used on a 0–5 number line, except the assessor modeled the fraction 7/2, and the students were asked to locate 3/2 as the practice fraction. 0–5 number line test items included both proper and improper fractions: 7/8, 11/7, 13/4, 9/5, 13/6, 7/3, 10/3, 9/2, 19/4, and 1/5. Examples of items are presented in Fig. 1.

Because performance on the three number lines (0–1, 0–2, and 0–5) was highly correlated, we combined the 38 items to create one fractions number line estimation score ($\alpha = .96$).

The score on this measure was calculated as the absolute value of the difference between the estimated position and actual position, divided by the numerical range of the number line (1, 2, or 5), multiplied by one hundred. A higher percent absolute error (PAE) score indicates poorer performance on the task.

4.3.3. **Fraction procedures**

The fraction procedures measure included 26 fraction computation items (adapted from Hecht, 1998). The set of items included six addition (e.g., 3 3/8 + 1 2/8), six subtraction (e.g., 1 1/3 – 4/5), nine multiplication (e.g., 5/6 × 3/4), and five division (e.g., 2 ÷ 3/4) problems. Seven problems included mixed numbers, and eight problems had like denominators. The problems, presented horizontally, were: 2/5 + 1/5, 3/4 – 1/4, 3/6 + 1/6, 5/6 – 2/6, 1 1/4 – 1/4, 3/4 + 2/4, 3 3/8 + 1 2/8, 2 2/3 – 1 1/3, 5/6 + 2/3, 7/8 – 1/2, 1 1/3 – 4/5, 3/4 + 2/3, 3 × 1/3, 40 × 1/2, 4 × 4/5, 6 × 3/4, 7/8 × 2/5, 5/6 × 3/4, 2 2/3 × 1/2, 1 3/8 × 2/3, 2 1/3 × 3 3/8, 1 1/3 ÷ 4, 1/6 ÷ 3, 2 ÷ 3/4, 7 ÷ 1/2, and 3/4 ÷ 1/8. Internal reliability was .82 in sixth grade.

4.4. **Procedure**

The predictor variables were assessed in fifth grade. Whole number line estimation, division, working memory (WMTHB-C), and attention (SWAN) were assessed in winter of fifth grade; non-symbolic proportional reasoning, multiplication fluency (WIAT), and reading fluency (TOWRE) were assessed in spring of fifth grade (we needed to use two measurement time points due to practical constraints in the schools). The fraction outcome measures were assessed in sixth grade. The fraction concepts
and procedures outcome measures were given in a whole class setting, and the fraction number line estimation task was administered individually in winter of sixth grade. With the exception of whole number division and multiplication fact fluency (which were paper and pencil tasks administered in a whole group setting), all predictor measures were administered individually. Trained assessors read instructions for each task verbatim. In all analyses, we controlled for the background variables of age, gender, special education status (as determined by the school district), income status (as determined by participation in free/reduced lunch program at school), and English learner status.

5. Results

Correlations among all variables are shown in Table 2. Statistically significant bivariate correlations were present among all general and math-related predictor and outcome variables. Mean scores on all measures for participants are presented in Table 3 (although for context we present percentile scores for the standardized norm-referenced measures, raw scores were used for all analyses).

To investigate the relative importance of general cognitive and math-related skills in predicting outcome measures, we used two direct-entry multiple regression analyses (MRAs) with fraction concepts and fraction procedures as the dependent variables, respectively. Initially, we had planned to include fraction number line estimation separately as a dependent variable. However, fraction number line estimation was highly correlated with the general fraction concepts measure ($r = .730$) (the correlation remained equally high, $r = .72$ when we removed the 5 NAEP items that involved number lines from the fraction concepts measure). Preliminary analyses showed that when fraction number line estimation was used as a dependent variable, the pattern of results concerning unique variance was very similar to the results of the analysis using general fraction concepts as a dependent variable. Because the NAEP test is well known to practitioners and policymakers, we report the regression analysis with general fraction concepts rather than fraction number line estimation.

In addition to the predictor variables of whole number estimation, working memory, non-symbolic proportional reasoning, attention, multiplication fact fluency, and long division, we controlled for demographic variables of age, gender (1 = female), special education status (1 = receiving), English learner (1 = yes), and income status (1 = low income). Because the non-symbolic proportional reasoning items with discrete markings were highly correlated with those that were continuous ($r = .83$) and the results of preliminary analyses did not change when we put them in our models separately, the proportional reasoning measure that was comprised of both types of items.

The MRA predicting fraction concepts was statistically significant, $R^2 = .577, F(12, 321) = 36.544, p = .001$. Table 4 shows the unstandardized and standardized beta coefficients, standard errors, and $R$-square change for each predictor variable, with all others controlled. Five predictors made unique and significant contributions to the estimation of fraction concepts: whole number line estimation, non-symbolic proportional reasoning, attention, working memory, and division. The relative contribution of the predictor variables was evaluated through the comparison of standardized beta coefficients, which serve as effect sizes (Cohen, Cohen, West, & Aiken, 2003; Keith, 2006; Pedhazur, 1997). It is appropriate to examine the standardized beta weights to determine the contribution of each predictor in the regression equation while holding the other predictors constant (Nathans, Oswald, & Nimmons, 2012). Whole number line estimation made the largest unique contribution ($\Delta R^2 = .079$); it was 2.1 times larger than that of attention ($\Delta R^2 = .170$), 1.8 times larger than that for non-symbolic proportional reasoning ($\Delta R^2 = .198$), and more than three times the size for working memory ($\Delta R^2 = .114$) and division ($\Delta R^2 = .113$).

The overall association of the MRA predicting fraction procedures (Table 5) was also significant although the variance explained by the predictors ($R^2 = .395, F(12, 321) = 17.451, p = .001$) was substantially smaller than for fraction concepts (which was 1.5 times larger; $R^2 = .577/395)$. Four predictors made unique contributions to the estimation of fraction procedures: attention, whole number line estimation, multiplication fact fluency, and long division. Unlike fraction concepts, attention made the largest unique contribution ($\Delta R^2 = .042$) in the fraction procedures model. The standardized beta weight was 1.5 times larger than that for multiplication fluency and long division, respectively ($\Delta R^2 = .171$ and $\Delta R^2 = .171$) and 1.4 times larger than that for whole number line estimation ($\Delta R^2 = .183$). Background variables and reading fluency did not contribute significantly to either model.

6. Discussion

We examined the relative importance of a constellation of math-related and cognitive predictors to knowledge of fraction concepts and fraction procedures in sixth grade. Children who leave sixth grade with weak fraction knowledge are likely to experience even greater mathematics problems in subsequent grades (Mazzocco & Devlin, 2008). Together, the predictors explained about 58% of the variance in fraction concepts; independently important were whole number line estimation, non-symbolic proportional reasoning, attention, working memory, and long division. For fraction procedures, the predictors explained about 40% of the variance in performance, with attention, number line estimation, multiplication fact fluency, and long division all making independent contributions. Background variables did not uniquely contribute to the regression models nor did reading fluency. Although the predictors investigated in this study accounted for a relatively large proportion
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Correlations among all measures.

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</tr>
</tbody>
</table>

* * p < 0.05.
** p < 0.01.
Table 3  
Means of predictor and outcome variables for study participants (n = 334).  

<table>
<thead>
<tr>
<th>Variables</th>
<th>Mean</th>
<th>SD</th>
</tr>
</thead>
<tbody>
<tr>
<td>Whole number line estimation (PAE)</td>
<td>8.21</td>
<td>5.44</td>
</tr>
<tr>
<td>Non-symbolic proportional reasoning (48)</td>
<td>33.65</td>
<td>10.26</td>
</tr>
<tr>
<td>Long division (6)</td>
<td>3.83</td>
<td>1.80</td>
</tr>
<tr>
<td>Multiplication fact fluency (percentile)</td>
<td>55.39</td>
<td>28.06</td>
</tr>
<tr>
<td>Working memory (42)</td>
<td>20.73</td>
<td>4.45</td>
</tr>
<tr>
<td>Attention (63)</td>
<td>38.55</td>
<td>11.35</td>
</tr>
<tr>
<td>Reading fluency (percentile)</td>
<td>61.75</td>
<td>22.61</td>
</tr>
<tr>
<td>Fractions concepts (28)</td>
<td>21.48</td>
<td>4.89</td>
</tr>
<tr>
<td>Fractions procedures (26)</td>
<td>11.78</td>
<td>5.11</td>
</tr>
<tr>
<td>Fraction number line estimation (PAE)</td>
<td>14.17</td>
<td>9.04</td>
</tr>
</tbody>
</table>

*Note: All scores are raw scores unless indicated otherwise. Total possible points for raw scores are indicated in parentheses.

Table 4  
Results of multiple regression for fifth grade predictors of fraction concepts in winter of sixth grade.  

<table>
<thead>
<tr>
<th>Variable</th>
<th>B</th>
<th>SE B</th>
<th>( \beta )</th>
<th>( \Delta R^2 )</th>
<th>Bivariate correlation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Age</td>
<td>-.056</td>
<td>.034</td>
<td>-.063</td>
<td>.003</td>
<td>-.282</td>
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<tr>
<td>Gender</td>
<td>.650</td>
<td>.377</td>
<td>.066</td>
<td>.004</td>
<td>.045</td>
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<tr>
<td>Special education</td>
<td>-.471</td>
<td>.767</td>
<td>-.026</td>
<td>.000</td>
<td>-.353</td>
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<tr>
<td>English learner</td>
<td>-.956</td>
<td>.578</td>
<td>-.061</td>
<td>.004</td>
<td>-.076</td>
</tr>
<tr>
<td>Low income</td>
<td>.153</td>
<td>.378</td>
<td>.016</td>
<td>.000</td>
<td>-.193</td>
</tr>
<tr>
<td>Whole number line estimation</td>
<td>-.324</td>
<td>.042</td>
<td>-.361**</td>
<td>.079</td>
<td>-.619</td>
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<tr>
<td>Working memory</td>
<td>.125</td>
<td>.046</td>
<td>.114**</td>
<td>.010</td>
<td>.437</td>
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<tr>
<td>Non-symbolic proportional reasoning</td>
<td>4.520</td>
<td>.943</td>
<td>.198***</td>
<td>.030</td>
<td>.491</td>
</tr>
<tr>
<td>Reading fluency</td>
<td>.024</td>
<td>.021</td>
<td>.048</td>
<td>.002</td>
<td>.333</td>
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<tr>
<td>Attention</td>
<td>.073</td>
<td>.019</td>
<td>.170***</td>
<td>.019</td>
<td>.504</td>
</tr>
<tr>
<td>Multiplication fact fluency</td>
<td>.025</td>
<td>.032</td>
<td>.037</td>
<td>.001</td>
<td>.419</td>
</tr>
<tr>
<td>Long division</td>
<td>.306</td>
<td>.138</td>
<td>.113</td>
<td>.007</td>
<td>.519</td>
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<tr>
<td>Constant</td>
<td>17.553</td>
<td>4.196</td>
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</tr>
</tbody>
</table>

*Note: \( R = .760, R^2 = .577 \) (N = 334, \( p < .001 \)).  
* \( p < .05 \).  
** \( p < .01 \).  
*** \( p < .001 \).

Table 5  
Results of multiple regression for fifth grade predictors of fraction procedures in winter of sixth grade.  

<table>
<thead>
<tr>
<th>Variable</th>
<th>B</th>
<th>SE B</th>
<th>( \beta )</th>
<th>( \Delta R^2 )</th>
<th>Bivariate correlation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Age</td>
<td>.008</td>
<td>.043</td>
<td>.009</td>
<td>.000</td>
<td>-.154</td>
</tr>
<tr>
<td>Gender</td>
<td>.486</td>
<td>.471</td>
<td>.047</td>
<td>.002</td>
<td>.051</td>
</tr>
<tr>
<td>Special education</td>
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<td>.960</td>
<td>.039</td>
<td>.001</td>
<td>-.233</td>
</tr>
<tr>
<td>English learner</td>
<td>.401</td>
<td>.723</td>
<td>.025</td>
<td>.001</td>
<td>-.035</td>
</tr>
<tr>
<td>Low income</td>
<td>-.137</td>
<td>.473</td>
<td>-.013</td>
<td>.000</td>
<td>-.150</td>
</tr>
<tr>
<td>Whole number line estimation</td>
<td>-.172</td>
<td>.052</td>
<td>-.183***</td>
<td>.020</td>
<td>-.408</td>
</tr>
<tr>
<td>Working memory</td>
<td>.012</td>
<td>.057</td>
<td>.011</td>
<td>.000</td>
<td>.309</td>
</tr>
<tr>
<td>Non-symbolic proportional reasoning</td>
<td>1.910</td>
<td>1.180</td>
<td>.080</td>
<td>.005</td>
<td>.343</td>
</tr>
<tr>
<td>Reading fluency</td>
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<td>.027</td>
<td>.011</td>
<td>.000</td>
<td>.275</td>
</tr>
<tr>
<td>Attention</td>
<td>.114</td>
<td>.024</td>
<td>.254**</td>
<td>.042</td>
<td>.500</td>
</tr>
<tr>
<td>Multiplication fact fluency</td>
<td>.119</td>
<td>.040</td>
<td>.171*</td>
<td>.017</td>
<td>.466</td>
</tr>
<tr>
<td>Long division</td>
<td>.486</td>
<td>.172</td>
<td>.171*</td>
<td>.015</td>
<td>.495</td>
</tr>
<tr>
<td>Constant</td>
<td>.725</td>
<td>5.250</td>
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</tr>
</tbody>
</table>

*Note: \( R = .628, R^2 = .395 \) (N = 334, \( p < .001 \)).  
* \( p < .05 \).  
** \( p < .01 \).  
*** \( p < .001 \).
of variance (especially for fraction concepts), unobserved variables explaining the remaining variance may include instructional influences, executive functions, and general intellectual capacity.

Generally, our findings replicate what has been observed in samples of younger children (e.g., Vukovic et al., 2014; Jordan et al., 2013) with sixth graders and also extend them by examining the influence of a wider range of math-related predictors – presumably malleable skills that have instructional relevance. Although the relative contribution of the predictors differed with the specific fraction outcome, number line estimation acuity continues to be important to the development of fraction knowledge in sixth grade. Children who estimate whole numbers on a number line with more precision seem to have an advantage in learning fractions. As students develop deeper fraction knowledge, they must see that properties true of whole numbers are not necessarily true of rational numbers. For example, each whole number has a unique predecessor and successor and therefore can be counted, whereas the infinite number of fractions cannot be counted directly (though students could count fractions with denominators that are the same units; e.g., 1/4, 2/4, 3/4, 4/4). Whole numbers never decrease when multiplied, while multiplying fractions can yield products smaller than either of the multiplicands. However, a property that unites fractions and whole numbers is that all real numbers are magnitudes that can be ordered along a continuous line. Understanding that whole numbers have magnitudes and can be placed on number lines is likely to help students develop this same understanding with fractions.

Students also may be thinking about part-whole relations and partitioning to arrive at their answer on the whole number line estimation task (e.g., to locate 250 on a number line, students might divide the line into quarters). In an eye tracking study of adults’ performance on a number line estimation task, Sullivan, Juhasz, Slattery, and Barth (2011) found that adults responded quickly; however, they also tended to fixate on the midpoint of the line. This suggested that they might have quickly used landmarks (e.g., 50 on a 0–100 line) to increase accuracy. These findings are in line with the position of Barth and Paladino (2011), who argue that the whole number line estimation task may be reflecting proportional reasoning skill.

By sixth grade, whole number line estimation is more important to fraction concepts than to procedures, which contrasts with the previous finding that third-grade number line estimation is equally predictive of fraction concepts and procedures at fourth grade (Jordan et al., 2013). One explanation for this developmental difference might be that fourth grade fraction arithmetic involves relatively simple procedures involving like denominators for addition and subtraction; by sixth grade, students are expected to be competent with multiple procedures and operations. For fraction procedures, attention and whole number computational abilities (multiplication fluency and division) emerge as key predictors, as has been noted in previous research (e.g., Hecht & Vagi, 2010).

A new and interesting finding is that performance on a visually presented, non-symbolic proportional reasoning measure contributes independently to students’ proficiency with fraction concepts, over and above symbolic math-related (i.e., number line estimation) and general predictors. The finding is in keeping with Möhring et al. (2013), who showed that children’s proportional reasoning is related to the ability to compare fraction magnitudes. Although proportional reasoning and whole number line estimation are correlated ($r = 0.37$), the independent contribution of the proportional reasoning task suggests that understanding scale relations and multiplicative reasoning (e.g., 1/3 is the same as 3/9; Boyer & Levine, 2012; Gunderson, Ramirez, Beilock, & Levine, 2012) may be independently important to fraction concepts, especially in the context of the widely used area model for representing fractions.

Is the non-symbolic proportional reasoning task tapping into spatial abilities more generally? Bailey et al. (2014) found that general spatial abilities in first grade do not predict fraction knowledge in sixth grade, after controlling for other general and math-specific cognitive processes. Additionally, Jordan et al. (2013) found that third grade general nonverbal abilities, as measured by a non-symbolic matrix completion task, do not independently predict fraction knowledge in fourth grade over and above number-specific processes. Vukovic et al. (2014) also found that nonverbal abilities in first grade were not associated with fraction concepts in fourth grade. In the present study, the proportional reasoning task was given after children had received some fractions instruction in elementary school. Such instruction could influence students’ approach to the proportional reasoning task, especially on items with discrete markings, which might encourage students to count the parts of the red “juice”
portion as well as the whole amount (blue water plus red juice) and then to convert into a ratio or fraction. However, children’s skill with marked items was highly related to their skill on the non-marked items and the mean response time was fast for both conditions (although the mean of 3.47 s for the continuous items was about a half of a second faster than for the discrete items). These results indicate that children probably were not counting or converting to a conventional algorithm. It is possible that the non-symbolic proportional reasoning task is measuring more targeted spatial abilities (e.g., scaling) that are important for fractions, which might explain why more general spatial tasks (as well as non-symbolic approximate number skills on an ANS task; Jordan et al., 2013) have not been found to be predictive of fraction skills. Future research should investigate the trajectory of proportional reasoning development in early elementary school and whether it predicts later fraction skills.

Long division skill was a small but unique predictor for both outcome measures. This finding contradicts the common assumption that skill with long division primarily reflects procedural learning. Long division emphasizes the place value of numbers and integration of operations, which also may be useful to fraction understanding (Siegler & Pyke, 2013). In contrast, multiplication fluency was uniquely predictive fraction procedures but not concepts, which makes sense given the importance of multiplication for finding common denominators (e.g., 3/4+2/3). Taken as a set, the importance of these number skills suggest that the whole number bias many students demonstrate when they encounter fraction computation problems might be easier to overcome for students with a strong grasp of whole numbers.

Of the set of more general predictors we examined, attentive behavior was distinctly predictive of fraction outcomes, over and above the contributions of the others. Attentive students benefit from classroom mathematics instruction more than attentive students (e.g., Hecht et al., 2003; Jordan et al., 2013). Working memory explained unique variance in fractions concepts, but not fraction procedures. In previous work, however, counting recall predicted fraction procedures in fourth grade (Jordan et al., 2013). Moreover, Seethaler, Fuchs, Star, and Bryant (2011) found that working memory, as assessed by a backwards digit recall task, accounted for unique variance in rational number computation skill in fifth graders after controlling for other domain general cognitive processes. Further research is needed to explore the relationship between working memory and later fraction outcomes. It is possible that executive processes other than working memory are more important for fraction learning, especially as students become more advanced. For example, some fraction items may present particular challenges for inhibitory processes. A look at errors on the fraction number line estimation task demonstrate that shifting from the 0 to 1 to 0 to 2 and 0 to 5 line ranges presents particular challenges for inhibitory processes. Children tended to look at a fraction of the entire line, even on fractions less than one. For example, when asked to locate 1/2 on a 0–2 number line, many children placed the mark in the middle of the whole line, which would be 1/2 of 2 but not the actual location of 1/2 on the number line (which is 1/2 of 1).

Overall, the findings show that a constellation of cognitive processes independently support the development of fractions understanding. The data show that the ability to represent the magnitudes of whole numbers foreshadows students’ ability to represent the magnitudes of fractions, supporting Siegler and colleagues’ empirical findings (e.g., Bailey, Siegler, & Geary, 2014) as well as their integrated theory of numerical development (e.g., Siegler et al., 2011; Siegler & Lortie-Forgues, 2014). Symbolic (i.e., number line estimation) and non-symbolic magnitude representations (i.e., proportional reasoning) seem to be independently important. In keeping with Hecht et al.’s (2003) work, however, domain general processes, such as attention and working memory, also contribute to fraction development. Moreover, foundational math skills related to multiplication fact fluency and long division also play a significant role in fractions development. Thus, as is usually the case in cognitive development, both domain specific knowledge (in this case, knowledge of numerical magnitudes and procedures) and domain general knowledge (in this case, attention and working memory) shape the change process.

The potential importance of numerical magnitude estimation and proportional reasoning to fraction learning has implications for instructional research and practice. Students who develop an understanding that each real number, including each fraction, is assigned a specific location on the number line have an advantage in learning not only fractions but also algebraic concepts (Booth, Newton, & Twiss-Garitty, 2013). A fraction intervention that emphasized a number line or
“measurement” interpretation of fractions increased at-risk learners’ skill with fraction concepts and procedures more than standard classroom instruction that focused on parts of wholes (Fuchs et al., 2013). However, the independent contributions of number line estimation and proportional reasoning found in the present study suggest that both number line and parts of a whole approaches are important instructional components for understanding fractions. In keeping with the present findings, research also is beginning to show that instructional approaches or treatments that target students’ strengths and weaknesses in cognitive processes, such as working memory, influence their responses to fraction instruction (e.g., Fuchs et al., 2014).

The lower overall predictability of our measures for fraction procedures compared to fraction concepts suggests that instruction in fraction computation algorithms may be used more in current practice than instruction geared toward building a basic understanding of what fractions are. The findings also suggest that long division should be viewed as more than just a rote procedure. Long division seems to be important for helping students integrate their understanding of operations and place value, which may also help students understand fraction computation problems, although this issue needs further study.

As many states implement the Common Core State Standards (Council of Chief State School Officers & National Governors Association Center for Best Practices, 2010) fractions are receiving more attention in U.S. classrooms. For example, the CCSS recommend that by the end of sixth grade, students should show proficiency in fraction equivalence and ordering as well as in executing complex fraction arithmetic procedures for all four operations, to provide a foundation for more advanced mathematical topics (e.g., ratios, rate, algebra). The present study reveals the particular importance of numerical magnitude knowledge for acquiring this fraction knowledge. Additionally, researchers and practitioners should explicitly consider students’ attention, working memory, long division, and multiplication fact fluency skills when developing fractions interventions.

References


